In the winter of 2013-2014, I decided to write up complete solutions to the starred exercises in *Differential Topology* by Guillemin and Pollack. There are also solutions or brief notes on non-starred ones. Please email errata to ceur@college.harvard.edu.

**Notation:** A neighborhood is always assumed to be an open neighborhood. A graph of a function $f$ is denoted $\Gamma(f)$.

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1 Chapter 1: Manifolds and Smooth Maps

1.1 Definitions

Exercise 1 (1.1.2). If \( X \subset \mathbb{R}^N, Z \subset X, \) and \( f : X \to \mathbb{R}^m \) is smooth / diffeomorphic, then \( f|_Z \) is also smooth / diffeomorphic.

Solution) It suffices to show the smooth part (then apply it to the inverse map to get diffeomorphism). Fix any \( z \in Z \subset X. \) Since \( f : X \to \mathbb{R}^m \) is smooth, there exists \( z \in U \subset \mathbb{R}^N \) open (i.e. \( X \cap U \) a neighborhood of \( z \in X \)) and \( F : U \to \mathbb{R}^m \) smooth such that \( F = f \) on \( X \cap U \). Now, \( U \cap Z \) is a neighborhood of \( z \in Z \) such that \( F = f|_Z \) on \( Z \cap U. \) □

Exercise 2 (1.1.3). Let \( X \subset \mathbb{R}^N, Y \subset \mathbb{R}^M, Z \subset \mathbb{R}^L, \) and let \( f : X \to Y, g : Y \to Z. \) Then: \( f \) and \( g \) are smooth / diffeomorphism \( \Rightarrow \) \( g \circ f : X \to Z \) is smooth / diffeomorphism.

Solution) It suffices to show the smooth part (diffeomorphism part follows easily). Note that this is true when \( X, Y, Z \) are open subsets of \( \mathbb{R}^N, \mathbb{R}^M, \mathbb{R}^L \) (Chain Rule from calculus). Now fix any \( x \in X, y := f(x). \) We have \( y \in V \subset \mathbb{R}^M \) open with \( G : V \to \mathbb{R}^L \) smooth, and \( x \in U \subset \mathbb{R}^N \) open with \( F : U \to V \) smooth (if necessary by replacing \( U \) with \( U \cap f^{-1}(V) \)), and \( F = f \) on \( X \cap U, G = g \) on \( Y \cap V. \) Then \( G \circ F : U \to \mathbb{R}^L \) is smooth, and \( G \circ F = g \circ f \) on \( X \cap U \) since \( f(X \cap U) \subset (Y \cap V) \) by construction. □

Exercise 3 (1.1.4). Show that any open ball \( B_r(0) \) in \( \mathbb{R}^k \) is diffeomorphic to \( \mathbb{R}^k, \) and hence, if \( X \) is a \( k \)-dimensional manifold then every point in \( X \) has a neighborhood diffeomorphic to \( \mathbb{R}^k. \)

Solution) Consider the maps

\[
B_r(0) \to \mathbb{R}^k, \quad x \mapsto \frac{r x}{\sqrt{r^2 - \|x\|^2}} \quad \text{and} \quad \mathbb{R}^k \to B_r(0), \quad y \mapsto \frac{ry}{\sqrt{r^2 + \|y\|^2}}
\]

They are mutual inverses, and by the previous exercise both are smooth (lots of composition of smooth maps). Lastly, if \( x \in X \) and \( \phi : U \to X \) is local parametrization at \( x, \) then take (for small enough \( r > 0 \)) \( B_r(\phi^{-1}(x)) \subset U \) so that \( \phi|_{B_r} \) is also a local parametrization at \( x. \) With diffeomorphism \( \psi : \mathbb{R}^k \to B_r, \) we have a local parametrization \( \phi|_{B_r} \circ \psi : \mathbb{R}^k \to X \) at \( x. \) □

Exercise 4 (1.1.5). Every \( k \)-dimensional vector subspace \( V \) of \( \mathbb{R}^N \) is a manifold diffeomorphic to \( \mathbb{R}^k, \) all linear maps on \( V \) are smooth, and if \( \phi : \mathbb{R}^k \to V \) is a linear isomorphism, then the corresponding coordinate functions are linear functionals on \( V \) (called linear coordinates).

Solution) Lemma: every linear transformation \( L : \mathbb{R}^n \to \mathbb{R}^m \) is smooth (: \( dL_x = L \forall x \in \mathbb{R}^n \)), and note that any linear map on \( V \) is smooth since it extends to a linear map on \( \mathbb{R}^N. \)

Now, by choosing a basis of \( V \) we have an isomorphism \( \phi : V \to \mathbb{R}^k, \) which we can extend to a linear map \( \tilde{\phi} : \mathbb{R}^N \to \mathbb{R}^k. \) So, \( \phi \) is smooth, \( \phi^{-1} : \mathbb{R}^k \to V \subset \mathbb{R}^N \) (linear) is also smooth, and thus \( V \) is a manifold diffeomorphic to \( \mathbb{R}^k. \)

Let \( \phi := (x_1, \ldots, x_k). \) Since \( x_i = \pi_i \circ \phi, \) each \( x_i \) is a linear functional on \( V \) (projections \( \pi_i : \mathbb{R}^N \to \mathbb{R} \) is linear). □
Exercise 5 (1.1.6, 7, 8). Smooth bijection need not be diffeomorphism. Union of two coordinate axes in $\mathbb{R}^2$ is not a manifold; hyperboloid in $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = a$, $a > 0$ is a manifold (not when $a = 0$).

Solution) $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^3$ is smooth and bijective but $f^{-1}(x) = \sqrt[3]{x}$ is not smooth at $x = 0$. When $(0, 0)$ is removed from $\{x = 0\} \cup \{y = 0\}$ we get four disconnected components, but $\mathbb{R} - \{\ast\}$ has two components, and $\mathbb{R}^n - \{\ast\}$ is connected for $n \geq 2$. Lastly, if $H$ is the hyperboloid, $H^+$ and $H^-$ can be parametrized by $\mathbb{R}^2 - B_0(0) \to \mathbb{R}^3$ via $(u, v) \mapsto (u, v, \pm \sqrt{u^2 + v^2 - a})$, and for $(x, y, 0) \in H \cap \{z = 0\}$, it also has a local parametrization $B_0(0) \to \mathbb{R}^3$ via $(u, v) \mapsto (\pm \sqrt{a + u^2 - v^2}, u, v)$ (switch $x, y$ if necessary). When $a = 0$, $(0, 0, 0)$ has no neighborhood diffeomorphic to $\mathbb{R}^2$ (again use connectivity argument).

Exercise 6 (1.1.12,13). Let $S^k \subset \mathbb{R}^{k+1}$ sphere, and $N = (0, \ldots, 0, 1)$ the north pole. Then the stereographic projection $\pi : S^k - \{N\} \to \mathbb{R}^k$ is a diffeomorphism.

Solutions) Direct computation yields the maps explicitly:

$$
\pi : (u_1, \ldots, u_{k+1}) \mapsto \left( \frac{u_1}{1 - u_{k+1}}, \ldots, \frac{u_k}{1 - u_{k+1}} \right)
$$

$$
\pi^{-1} : (x_1, \ldots, x_k) \mapsto \left( \frac{2 x_1}{x_1^2 + \cdots + x_k^2 + 1}, \ldots, \frac{2 x_k}{x_1^2 + \cdots + x_k^2 + 1}, \sqrt{\frac{x_1^2 + \cdots + x_k^2 - 1}{x_1^2 + \cdots + x_k^2 + 1}} \right)
$$

and that $\pi, \pi^{-1}$ are smooth follows from [1.1.3] and that they are mutual inverses is checked by direct computation. (This also shows that $S^k$ is a $k$-dimensional manifold).

Exercise 7 (1.1.14,15). If $f : X \to X'$, $g : Y \to Y'$ smooth, then the product map $f \times g : X \times Y \to X' \times Y'$ defined by $(x, y) \mapsto (f(x), g(y))$ is smooth. Hence, if $f, g$ are diffeomorphisms, $f \times g$ is also a diffeomorphism. Lastly, projection map $X \times Y \to X$ is smooth.

Solution) Note that for $A, A' \subset X$ and $B, B' \subset Y$, $(A \times B) \cap (A' \times B') = (A \cap A') \times (B \cap B') \subset X \times Y$ (NOT true for unions). Also, it is easy to check that $f \times g$ is smooth when $X, Y$ are open subsets ($\cdot : f \times g = (f_1, \ldots, f_N, g_1, \ldots, g_M)$), so taking neighborhoods $U, V$ of $X, Y$ with $F, G$ smooth, $F \times G = f \times g$ on $(X \cap U) \times (Y \cap V)$.

If $f, g$ are diffeomorphisms, $f^{-1} \times g^{-1}$ is smooth inverse of $f \times g$. Finally, $X \times Y \to X$ is smooth since $\mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ is smooth (in fact the Jacobian looks like $[I_N \mid 0]$).

Exercise 8 (1.1.16,17). Let $f : X \to Y$ be smooth, and define $\tilde{f} : X \to \Gamma(f)$ by $x \mapsto (x, f(x))$. Then $\tilde{f}$ is a diffeomorphism (if $X$ a manifold so is $\Gamma(f)$).

Solution) Lemma: the diagonal map $\Delta : X \to X \times X$, $x \mapsto (x, x)$ is a diffeomorphism ($\cdot : \Delta : \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ has Jacobian $[I_N \mid I_N]^t$ so it is smooth; the inverse map is same as projection so it is smooth). Now, using the lemma and the previous exercise, $\tilde{f} = (\text{Id} \times f) \circ \Delta$ is smooth. The inverse map is same as the projection map, so it is smooth, so $\tilde{f}$ is a diffeomorphism.
Exercise 9 (1.1.18). Let $0 < a < b$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 
  e^{-1/x^2} & x > 0 \\
  0 & x \leq 0 
\end{cases}$$

is smooth. $g(x) := f(x - a)f(b - x)$ is a smooth map, and is positive on $(a,b)$ and zero elsewhere. Moreover, $h(x) := \int_{x}^{\infty} gdx$ is also smooth with the property that $h(x) = 0$ for $x \leq a$, $h(x) = 1$ for $x \geq b$, and $0 < h'(x) < 1$ for $x \in (a,b)$ ($h$ is also non-decreasing). Lastly, construct a smooth function $H : \mathbb{R}^k \to \mathbb{R}$ that is 1 on $B_a$, 0 on $\mathbb{R}^k - B_b$, and $(0,1)$ otherwise.

Solution) $f$ is smooth: for $x > 0$ the $i$th derivative of $f$ is of the form (rational polynomial)$e^{-1/x^2}$ hence $\lim_{x \to 0} f^{(i)}(x) = 0, \forall i = 0, 1, \ldots$. Addition/multiplication by a constant and $(x,y) \mapsto xy$ are smooth maps, so $g$ is smooth with desired property. Lastly, $h$ is smooth with desired property by Fundamental Theorem of Calculus. Now note that $\mathbb{R}^k \to \mathbb{R}, x \mapsto ||x||$ is smooth, so $x \mapsto 1 - h(||x||)$ is the desired function $H$. □

1.2 Derivatives and Tangents

Note: When the context is clear, $X, Y$ are assumed to be manifolds (of dimension $\mathbb{R}^k, \mathbb{R}^l$ residing in $\mathbb{R}^N, \mathbb{R}^M$).

Exercise 10 (1.2.1,2). Let $X \subset Y$ be a submanifold, and $j : X \hookrightarrow Y$ be the inclusion map. Then $\forall x \in X$, $dj_x : T_x(X) \to T_x(Y)$ is injective—in fact it is an inclusion. If $U$ is an open subset of a manifold $X$, $T_x(U) = T_x(X)$ for $x \in U$.

Solution) Lemma: if $X$ is a manifold, $x \in X$, and $\phi : U \to X$ is a local parametrization with $\phi(0) = x$, then $(d\phi_0)^{-1} = d(\phi^{-1})_{x}$ (∴ Exercise [1.2.4]). Now, let $x \in X$, $\phi : U \to X$, $\psi : V \to Y$ be local parametrizations ($U \subset \mathbb{R}^k, V \subset \mathbb{R}^l$ open, $\phi(0) = x = \psi(0)$). Note that $\psi^{-1} \circ j \circ \phi = \psi^{-1} \circ \phi$, and so we have $dj_x = d\psi_0 \circ d(\psi^{-1} \circ \phi)_0 \circ (d\phi_0)^{-1} = d\psi_0 \circ d(\psi^{-1})_y \circ d\phi_0 \circ (d\phi_0)^{-1} = \text{Id}$ (∴ first equality by definition, second by chain rule, third by lemma). □

Exercise 11 (1.2.3+α). Let $V \subset \mathbb{R}^N$ be a vector subspace. Then $\forall x \in V$, $T_x(V) = V$. Moreover, if $L : V \to \mathbb{R}^M$ is a linear map, then for $x \in V$, $dL_x = L$.

Solution) By choosing basis we have linear isomorphism $\phi : \mathbb{R}^k \to V$, and $\phi = d\phi_u$ for $\phi(u) \in V$, hence $T_x(V) = V$. The remaining statement follows from [Exercise 17]; extend $L$ to a linear map on the whole $\mathbb{R}^N$, then apply [Exercise 17] and $T_x(V) = V$. □

Exercise 12 (1.2.4). Let $f : X \to Y$ is a (local) diffeomorphism and $x \in X, y := f(x)$, then the linear map $df_x : T_x(X) \to T_y(Y)$ is an isomorphism. In fact, $(df_x)^{-1} = df(f^{-1})_y$.

Solution) Let $f^{-1}$ be the smooth (local) inverse map. Obviously, $d\text{Id} = \text{Id}$, so from the chain rule we have $\text{Id} = d(f \circ f^{-1})_y = df_x \circ d(f^{-1})_y$, and likewise, $d(f^{-1})_y \circ df_x = \text{Id}$. □
Exercise 13 (1.2.8). What is the tangent space to the hyperboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$ $(a > 0)$?

Solution) As in [1.1.8], we have a local parametrization $\phi : B_a(0) \to \mathbb{R}^3$ given by $(u, v) \mapsto (\sqrt{a + u^2 - v^2}, u, v)$, and $[d\phi] = \begin{bmatrix} u(a + u^2 - v^2)^{-1/2} & -v(a + u^2 - v^2)^{-1/2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. And so $[d\phi(\sqrt{a}, 0, 0)] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, and thus the tangent space is the $y, x$-plane, as expected. □

Exercise 14 (1.2.9). Let $X, Y$ manifolds, $\pi : X \times Y \to X$ be the projection map, and for $y \in Y$ let $f^y : X \to X \times Y$ be a smooth injection defined by $x \mapsto (x, y)$. Then

1. $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$
2. $d\pi_{(x,y)} : T_x(X) \times T_y(Y) \to T_x(X)$ is also a projection $(v, w) \mapsto v$.
3. $f^y$ is diffeomorphism onto its image and $d(f^y)_x : v \mapsto (v, 0)$.

Lastly, if $f : X \to X'$, $g : Y \to Y'$ are smooth maps, then $d(f \times g)(x, y) = df_x \times dg_y$.

Solution) Let $(x, y) \in X \times Y$ and $\phi \times \psi : U \times V \to X \times Y$ be local parametrization. Now (1.) follows from the following lemma:

Lemma: Let $U_1, U_2$ be open subset of $\mathbb{R}^n, \mathbb{R}^m$, and $g_1 \times g_2 : U_1 \times U_2 \to \mathbb{R}^n \times \mathbb{R}^m$ is smooth, then $dg_{(u_1, u_2)} = d(g_1)_{u_1} \times d(g_2)_{u_2}$ (as $[dg_{(u_1, u_2)}] = \begin{bmatrix} [d(g_1)_{u_1}] & 0 \\ 0 & [d(g_2)_{u_2}] \end{bmatrix}$). For (2.),

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\pi} & X \\
\phi \times \psi & \uparrow & \phi \\
U \times V & \xrightarrow{h} & U
\end{array}
$$

we note that the map $h := \phi^{-1} \circ \pi \circ (\phi \times \psi) : U \times V \to U$ is also the projection map $(u, v) \mapsto u$. So, $d\pi_{(x,y)} = d\phi_0 \circ dh_{(0,0)} \circ d(\phi \times \psi)^{-1}_0$ is easily computed to be the projection, as desired. Now for (3.), $f^y$ is diffeomorphism since $(f^y)^{-1}$ is just projection, and we have the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f^y} & X \times y \subset X \times Y \\
\phi & \uparrow & \phi \times \psi \\
U & \xrightarrow{h} & U \times 0 \subset U \times V
\end{array}
$$

It’s easy to see that $dh_0 = \text{Id}_k \times 0$. So, $d(f^y)_x(v) = (d\phi_0 \times d\psi_0) \circ d\phi_0^{-1}(v) = (v, 0)$, as desired.

Finally, we show $d(f \times g)(x, y) = df_x \times dg_y$. Let $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ be projections, and $j_1 : X' \to \mathbb{R}^{N+M'}$, $j_2 : Y' \to \mathbb{R}^{N'+M'}$ be smooth maps $x' \mapsto (x', 0)$, $y' \mapsto (0, y')$. Now, note that $f \times g = j_1 \circ f \circ \pi_1 + j_2 \circ g \circ \pi_2$. So, using (1., 2., 3.) and chain rule, we have $d(f \times g)(x, y)(u, v) = (df_x(u), dg_y(v))$, as desired. □
Exercise 15 (1.2.10,11). Let \( f : X \to Y \) smooth map of manifolds and define \( \tilde{f} : X \to X \times Y \) by \( x \to (x, f(x)) \), then \( df_x(v) = (v, df_x(v)) \), and hence \( T_{(x,f(x))}(\Gamma(f)) \) is the graph of \( df_x : T_x(X) \to T_{f(x)}(Y) \).

Solution) Lemma: if \( \Delta : X \to X \times X \) is the diagonal map, then \( d\Delta_x(v) = (v,v) \) \( (\therefore h : U \to U \times U \) has \( dh_0 = [I_k | I_k] \)). Now, \( \tilde{f} = (\text{Id} \times f) \circ \Delta \), so the lemma and [Exercise 1.2.9] implies that \( \tilde{f}_x(v) = (v, df_x(v)) \), as desired. \( \square \)

Exercise 16 (1.2.12). A curve on a manifold \( X \) is a smooth map \( c : I \to X, t \mapsto c(t) \), where \( I \subset \mathbb{R} \) interval. The velocity vector of the curve \( c \) at time \( t_0 \), denoted \( dc_{t_0}/dt \) or \( \dot{c}(t_0) \), is defined as \( dc_{t_0}(1) \in T_{x_0}(X) \) where \( x_0 = c(t_0) \). Note that when \( X = \mathbb{R}^k \) and \( c(t) = (c_1(t), \ldots, c_k(t)) \), then \( dc_0 = (c'_1(t_0), \ldots, c'_k(t_0)) \). Every vector in \( T_x(X) \) is the velocity vector of some curve in \( X \), and conversely.

Solution) First, it is obvious that tangent space at a point on an interval is \( \mathbb{R} \), so \( dc_{t_0} : \mathbb{R} \to T_{x_0}(X) \), and we thus we have the converse.

Now fix any \( x \in X \) and \( v \in T_x(X) \). Let \( \phi : U \to X \) be local parametrization around \( x \), \( \phi(0) = x \), and WLOG \( U = \mathbb{R} \) (by [Exercise 1.1.4]). By definition \( v = d\phi_0(w) \) for some \( w = (w_1, \ldots, w_k) \in \mathbb{R}^k \). Now, define \( \tilde{c} : \mathbb{R} \to \mathbb{R}^k \) by \( t \mapsto (w_1t, \ldots, w_kt) \), and \( c := \phi \circ \tilde{c} \). Now, \( dc_0 = d\phi_0 \circ \tilde{c}_0 \) and so \( dc_0(1) = d\phi_0(w) \), as desired. \( \square \)

Exercise 17 (made-up). Let \( f : X \to Y \) be smooth map of manifolds, \( x \in X \), and let \( x \in U \subset \mathbb{R}^N \) open and \( F : U \to \mathbb{R}^M \) smooth such that \( F = f \) on \( X \cap U \). Then \( df_x = dF_x|_{T_x(X)} : T_x(X) \to T_y(Y) \).

Solution) WLOG, we have \( \phi : \tilde{U} \to X \cap U \) and \( \psi : \tilde{V} \to Y \cap V \) local parametrizations (diffeomorphisms) such that the following commutes (with \( h := \psi^{-1} \circ f \circ \phi, \phi(u) = x, \psi(v) = f(x) \))

\[
\begin{array}{ccc}
X \cap U & \xrightarrow{f} & Y \cap V \\
\phi \uparrow & & \uparrow \psi \\
\tilde{U} & \xrightarrow{h} & \tilde{V}
\end{array}
\]

Now since \( f = F \) on \( X \cap U \), \( h = \psi^{-1} \circ F \circ \phi \), and hence \( dh_u = d\psi_u^{-1} \circ dF_x \circ d\phi_u \), we have \( df_x = Id_{\mathbb{R}^M} \circ dF_x \circ Id_{T_x(X)} \), and hence \( df_x = dF_x|_{T_x(X)} \) as desired. In fact, \( dF_x \) is a linear extension of the linear map \( df_x \). \( \square \)

1.3 The Inverse Function Theorem and Immersions

Exercise 18 (1.3.2). Suppose \( Z \subset X \) \( l \)-dimensional submanifold, \( z \in Z \). Then there exists a local coordinate system \( \{x_1, \ldots, x_k\} \) on a neighborhood \( U \) of \( z \) in \( X \) such that \( Z \cap U \) is defined by the equations \( x_{l+1}, \ldots, x_k = 0 \).
Solution) Since the inclusion map \( j : Z \hookrightarrow X \) is an immersion, by Local Immersion Theorem, we have parametrizations around \( z \) such that:

\[
\begin{array}{ccc}
Z & \xrightarrow{j} & X \\
\phi \uparrow & & \psi \\
V & \text{can. imm.} & \bar{U}
\end{array}
\]

commutes, with \( \phi(0) = z = \psi(0) \) and \( \bar{U} = V \times V' \). Now, let \( \psi(\bar{U}) := U \subset X \) and \( \psi^{-1} = (x_1, \ldots, x_k) \), then \( Z \cap U = \{v \in U \mid x_{i+1}(v) = 0, \ldots, x_k(v) = 0\} \), as desired. \( \square \)

**Exercise 19** (1.3.3,4,5). If \( f : \mathbb{R} \to \mathbb{R} \) is a local diffeomorphism, then \( \text{Im}(f) \) is an open interval and \( f : \mathbb{R} \to \text{Im}(f) \) is a diffeomorphism. Such is not true for \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) local diffeomorphism.

Solution) Lemma[1.3.5]: if \( f \) is injective local diffeomorphism, its image is open in \( Y \), and it is diffeomorphism onto its image (\( \cdot \) : open: local homeomorphisms are open maps, and bijective local diffeomorphism admits smooth inverse ). \( \mathbb{R} \) is connected thus so is \( \text{Im}(f) \), hence \( \text{Im}(f) \) is an interval, and \( f \) local diffeomorphism implies that \( f \) is an open map and that \( f'(x) \neq 0 \) (and hence \( f \) is injective).

This is not true for \( \mathbb{R}^2 \); consider \( h = g \times \arctan \) where \( g : \mathbb{R} \to S^1 \), \( t \mapsto (\cos 2\pi t, \sin 2\pi t) \) (\( h \) is not injective). \( \square \)

**Exercise 20** (1.3.6). Let \( f : X \to Y, g : Y \to Y' \) be immersions, \( Z \subset X \) submanifold. Then \( f \times g, g \circ f, \) and \( f|Z \) are immersions, and if \( \dim X = \dim Y \), then \( f \) is in fact a local diffeomorphism.

Solution) \( f \times g \) is immersion by [Exercise 1.2.9] and \( g \circ f \) immersion by chain rule. If \( j : Z \hookrightarrow X \) is inclusion map, \( f|Z = f \circ j \), so \( f|Z \) is immersion. If \( \dim X = \dim Y \), \( f \) is local diffeomorphism by Local Immersion Theorem. \( \square \)

**Exercise 21** (1.3.7). Define \( g : \mathbb{R} \to S^1, t \mapsto (\cos 2\pi t, \sin 2\pi t) \) (local diffeomorphism) and \( G := g \times g : \mathbb{R}^2 \to S^1 \times S^1 \), and let \( L \subset \mathbb{R}^2 \) a line with irrational slope. Then \( G|_L \) is an injective local diffeomorphism, but its image is not a submanifold of \( S^1 \times S^1 \).

Solution) \( G : (s,t) \mapsto (\cos 2\pi s, \sin 2\pi s, \cos 2\pi t, \sin 2\pi t) \), and WLOG let \( L \) be defined by \( t = \alpha s, \alpha \in \mathbb{R} - \mathbb{Q} \). If \( G(s_1, \alpha s_1) = G(s_2, \alpha s_2) \), then \( s_1 - s_2 \in \mathbb{Z} \) and \( \alpha(s_1 - s_2) \in \mathbb{Z} \), which implies that \( s_1 = s_2 \) since \( \alpha \) is irrational. Moreover, since \( \{\alpha n\}_{n \in \mathbb{N}} \) dense in \( \mathbb{R}/\mathbb{Z} \), we have that \( \text{Im}(G|_L) \) is dense in \( S^1 \times S^1 \), so \( \text{Im}(G|_L) \) cannot be a submanifold. \( \square \)

**Exercise 22** (1.3.9). Let \( (x_1, \ldots, x_N) \) be standard coordinate functions on \( \mathbb{R}^N \), and \( X \subset \mathbb{R}^N \) be a \( k \)-dimensional manifold. Then any \( x \in X \) has a neighborhood on which the restrictions of some \( k \) coordinate functions \( x_{i_1}, \ldots, x_{i_k} \) form a local coordinate system.

Now for simplicity assume that \( x_1, \ldots, x_k \) form a local coordinate on a neighborhood \( V \) of \( x \in X \). Then \( \exists g_{k+1}, \ldots, g_N : U \subset \mathbb{R}^k \to \mathbb{R} \) smooth such that \( V = \Gamma(g) \) where \( g = (g_{l+1}, \ldots, g_N) : U \to \mathbb{R}^{N-k} \), and thus every manifold is locally a graph of a smooth function.
Solution) Choose a basis \( v_1, \ldots, v_k \in \mathbb{R}^N \) of \( T_x(X) \), then the matrix \( [v_1 \cdots v_k] \) has rank \( k \), so it has \( k \) linearly independent rows, say \( i_1, \ldots, i_k \). Now, let \( \pi : \mathbb{R}^N \to \mathbb{R}^k \) be projection defined by \((x_1, \ldots, x_N) \mapsto (x_{i_1}, \ldots, x_{i_k})\), then \( d\pi|_{T_x(X)} : T_x(X) \to \mathbb{R}^k \) is an isomorphism by construction, hence by IVT \( \pi|_X : X \to \mathbb{R}^k \) is a local diffeomorphism. Now, assume \( i_1, \ldots, i_k = 1, \ldots, k \) and let \( \bar{\pi} := \pi|_X \). \( \bar{\pi} : X \to \mathbb{R}^k \) is a diffeomorphism on \( x \in V \subset X \), and the smooth inverse \( \bar{\pi}^{-1} : U \to V \) is of form \( \text{Id} \times g \), and hence the result as desired. \( \square \)

**Exercise 23** (1.3.10). **Generalized IVT:** Let \( f : X \to Y \) be smooth map that is injective on a compact submanifold \( Z \subset X \), and suppose \( \forall x \in Z, df_x : T_x(X) \to T_{f(x)}(Y) \) is isomorphism. Then \( f \) maps \( Z \) diffeomorphically on \( f(Z) \), and in fact, maps an open neighborhood of \( Z \) in \( X \) diffeomorphically onto an open neighborhood of \( f(Z) \) in \( Y \). IFT is when \( Z \) is a single point.

**Solution** Since \( df_x \) is an isomorphism for all \( x \in Z \), for each \( x \in Z \) there exists \( U_x \) on which \( f|_{U_x} \) is a diffeomorphism. \( \{U_x\} \) is a cover of \( X \), hence we choose a finite subcover \( U = \bigcap U_i \supset Z \) (\( X \) is compact). On \( U \), \( f \) is a local diffeomorphism, so only need show that \( f \) is injective on some open set \( V \) containing \( Z \) (then \( f \) is injective local diffeomorphism on \( V \cap U \), hence a diffeomorphism).

Suppose \( V \) does not exist; then taking consecutively smaller \( \epsilon \)-neighborhoods \( Z^\epsilon \) of \( Z \), we obtain a sequence \( \{a_i\}, \{b_i\} \) such that \( a_i \neq b_i \) but \( f(a_i) = f(b_i) \). Passing through subsequences, \( a_i \to a \) and \( b_i \to b \) (both converge) since WLOG they all belong to some \( Z^\epsilon \) for which \( \bar{Z}^\epsilon \) is compact. Moreover, by construction the limit point is on \( Z \) and hence \( a = b = \bar{z} \) since \( f \) is injective on \( Z \). However, this implies that \( f \) cannot be a local diffeomorphism at \( \bar{z} \). Contradiction. \( \square \)

### 1.4 Submersions

**Exercise 24** (1.4.1). If \( f : X \to Y \) is a submersion and \( U \) is open in \( X \), then \( f(U) \) is open in \( Y \).

**Solution** Fix any \( y \in f(U) \); we need show \( \exists V \in Y \) open in \( Y \) such that \( V \subset f(U) \). Pick \( x \in f^{-1}(y) \), then by local submersion theorem we have the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\phi \uparrow & & \uparrow \psi \\
\bar{V} \times \bar{U} & \xrightarrow{\pi} & V \\
\end{array}
\]

with \( \phi(\bar{V} \times \bar{U}) \subset U \) and \( \psi(\bar{U}) := V \), and \( V \) is as desired. \( \square \)

**Exercise 25** (1.4.2). Let \( X \) compact and \( Y \) connected, then every submersion \( f : X \to Y \) is surjective; hence, there exist no submersions of compact manifolds into Euclidean spaces.

**Solution** Clearly, \( Y = f(X) \cup (Y - f(X)) \), so it suffices to show that \( Y - f(X) \) is both open and closed in \( Y \). \( f(X) \) is compact in \( Y \) since \( X \) is compact, hence \( f(X) \) is closed in \( Y \); \( X \) is open in \( X \), of by previous problem \( f(X) \) is open in \( Y \). Now, if \( f : X \to \mathbb{R}^m \) is a submersion then \( f \) is surjective, which is contradiction to \( \mathbb{R}^m \) not being compact. \( \square \)
Exercise 26 (1.4.5,6). Example: Let \( f : \mathbb{R}^3 \to \mathbb{R}, (x,y,z) \mapsto x^2 + y^2 - z^2, a,b \) both positive or negative. Then 0 is the only critical value of \( f \), and \( f^{-1}(a), f^{-1}(b) \) are diffeomorphic. More generally, let \( p \) be any homogeneous polynomial in \( k \) variables, then \( p^{-1}(a) (a \neq 0) \) is a \( k-1 \)-dimensional submanifold of \( \mathbb{R}^k \), and \( a > 0 \) ones are all diffeomorphic, as are \( a < 0 \) ones.

Solution) Lemma: \( p(x_1, \ldots, x_k) \) homogeneous of order \( m \), then \( \sum_{i=1}^{k} x_i \frac{\partial p}{\partial x_i} = m \cdot p \). The lemma implies that 0 is the only critical value of \( p \). Diffeomorphisms are made by scaling (note how \( m \) being odd or even makes a small difference). □

Exercise 27 (1.4.7). Suppose \( y \) is a regular value of \( f : X \to Y \), \( X \) compact, and \( \dim X = \dim Y \). Then \( f^{-1}(y) = \{ x_1, \ldots, x_n \} \) (finite), and there exists \( U \in y \) open in \( Y \) such that \( f^{-1}(U) = V_1 \cup \cdots \cup V_n \) where \( V_i \) is open neighborhood of \( x_i \) and \( f|_{V_i} : V_i \to U \) is a diffeomorphism \( (\forall i = 1, \ldots, n) \).

Solution) Note that \( y \) regular and \( \dim X = \dim Y \) implies that \( f \) is local diffeomorphism at any \( x \in f^{-1}(y) \). If \( f^{-1}(y) \) is not finite, it contains a limit point \( (\cdot : X \) compact), say \( x' \in f^{-1}(y) \), but \( f \) cannot be local diffeomorphism at \( x' \) (not injective). Now since \( f \) is local diffeomorphism at \( x_1, \ldots, x_n \), we can construct desired \( U \) and \( V_i \)'s by taking finite intersections. □

Exercise 28 (1.4.10,11). Tangent space to \( O(n) \) at identity \( I \) is the space of skew symmetric matrices. The group \( SL(n) \) is a (sub)manifold of \( M(n) \) and is moreover a Lie group, and the tangent space to \( SL(n) \) at the identity \( I \) is \( \{ H \in M(n) : \text{Tr}(H) = 0 \} \).

Solution) Recall \( O(n) = f^{-1}(I) \) where \( f : M(n) \to S(n), A \mapsto AA^t \). So, \( T_I(O(n)) = \ker df_I \), and since \( df_I : M(n) \to S(n), H \mapsto HH^t + HH^t = H^t + H, T_I(O(n)) = \{ H \in M(n) : H = -H^t \} \), as desired. \( SL(n) = \det^{-1}(1) \) where \( \det : M(n) \to \mathbb{R}, H \mapsto \det H, \) and 1 is a regular value of \( \det \) by [Exercise 1.4.6]. Moreover, matrix multiplication and inversion is a smooth map (Cramer’s rule), so \( SL(n) \) is a Lie group. Finally, we compute \( d \det_I : M(n) \to \mathbb{R} : \)

\[
\lim_{t \to 0} \frac{\det(I + tH) - \det(I)}{t} = \lim_{t \to 0} \frac{O(t^2) + \sum_{i=0}^{n} (tH)_{ii} + 1 - 1}{t} = \text{Tr}(H)
\]

and hence the desired result about tangent space to \( SL(n) \) at \( I \). □

Exercise 29 (1.4.13). (skip: this is linear algebra)

1.5 Transversality

Note: When the context is clear given \( x \) on \( X \) a manifold, we will not distinguish \( X \) as a whole manifold and the open neighborhood of \( x \) in \( X \) since the tangent space at \( x \) turns out same (will write \( X \) for both).

Exercise 30 (1.5.1,2). Examples of transversal & non-transversal intersection of linear spaces:

a. the \( xy \) plane and the \( z \)-axis in \( \mathbb{R}^3 \): transversal

b. the \( xy \) plane and the plane spanned by \( \{(3,2,0),(0,4,-1)\} \) in \( \mathbb{R}^3 \): transversal

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c. the plane spanned by \{(1,0,0),(2,1,0)\} and the y axis in \(\mathbb{R}^3\): not transversal

d. \(\mathbb{R}^k \times \{0\}\) and \(\{0\} \times \mathbb{R}^l\) in \(\mathbb{R}^{n}\): transversal if \(n \leq k + l\).

e. \(V \times \{0\}\) and the diagonal in \(V \times V\): transversal

f. symmetric and skew symmetric matrices in \(M(n)\): transversal

Solution) All of the above are easy to check with the following lemma:

Lemma: if \(V\) and \(W\) are linear subspaces of \(\mathbb{R}^n\), then \(V \cap W\) means \(V + W = \mathbb{R}^n\) (\(\because [\text{Exercise 1.2.3}])

**Exercise 31** (1.5.4). Let \(X\) and \(Z\) be transversal submanifolds of \(Y\), then for \(y \in X \cap Z\),

\[T_y(X \cap Z) = T_y(X) \cap T_y(Z)\]

Solution) We have \(V \subset Y\) open such that \(X \cap V = g^{-1}(0), Z \cap V = h^{-1}(0)\), for \(g = (g_1, \ldots, g_k) : V \to \mathbb{R}^k, h = (h_1, \ldots, h_l) : V \to \mathbb{R}^l\). Then \((X \cap Z) \cap V = f^{-1}(0)\) where \(f = (g \times h) \circ \Delta : V \to \mathbb{R}^{k+l}\) (N.B. 0 is regular value for \(f\) by transversality). Moreover, \(df_y : T_y(Y) \to \mathbb{R}^{k+l}, v \mapsto (dg_y(v), dh_y(v))\) (\(\because [\text{Exercise 1.2.3, 10}]\)). And since \(\ker df_y = \ker dg_y \cap \ker dh_y\), we have \(T_y(X \cap Z) = \ker df_y = \ker dg_y \cap \ker dh_y = T_y(X) \cap T_y(Z)\), as desired. □

**Exercise 32** (1.5.5). Let \(f : X \to Y, Z \subset Y\) submanifold, \(f \cap Z\), and \(W := f^{-1}(Z)\). Then \(T_x(W)\) is the preimage of \(T_{f(x)}(Z)\) under \(df_x : T_x(X) \to T_{f(x)}(Y)\), i.e. \(T_x(f^{-1}(Z)) = df_x^{-1}(T_{f(x)}(Z))\)

Solution) As in the proof, we have open neighborhoods \(U, V\) in \(X, Y\) around \(x, f(x)\) such that \(U \overset{f}{\to} V \overset{g}{\to} \mathbb{R}^l\) and \(Z \cap V = g^{-1}(0), f^{-1}(Z) \cap U = (g \circ f)^{-1}(0)\). Now, noting \(T_{f(x)}(Z) = \ker dg_{f(x)}\), we have \(T_x(Z) = \ker df_x = \ker dg_y \cap \ker dh_y\), as desired. (This implies [Exercise 1.5.4] \(f \circ i : X \to Y\) and \(di_x\) is just inclusion). □

**Exercise 33** (1.5.7). \(X \overset{f}{\to} Y \overset{g}{\to} Z\) smooth maps of manifolds, \(W \subset Z\) submanifold such that \(g \cap W\). Then \(f \cap g^{-1}(W)\) if and only if \((g \circ f)\cap W\).

Solution) Fix any \(x \in (g \circ f)^{-1}(W)\), and \(y := f(x), z := g(y)\). Note that since \(g \cap W\), \(dg_y(T_y(Y)) + T_x(W) = T_y(Z)\).

\((g \circ f)\cap W\) we need show \(\text{Im}(dg_y \circ df_x) + T_x(W) = T_y(Z)\). Let \(\tilde{w} \in T_x(Z)\) be given. \((g \cap W)\) implies \(\exists u \in T_y(Y), v \in T_x(W)\) such that \(dg_y(u) + v = \tilde{w}\). Moreover, \((g \cap W)\), so \(T_y(Y) = T_x(W) + (g^{-1}(W))\) \(= \text{Im}(dx_y) + T_y(T_y(Z))\) (\(\because [\text{Exercise 1.5.5}]\)), hence there exists \(\tilde{u} \in T_x(X)\) and \(v' \in T_y(Z)\) such that \(df_x(\tilde{u}) + v' = u\). Finally, we see that then \((dg_y \circ df_x)(\tilde{u}) + v' = \tilde{w}\), as desired.

\((g \circ f)\cap W\) we need show \(\text{Im}(df_x + dg_y^{-1}(T_x(W))) = T_y(Y)\). Again, fix a \(\tilde{w} \in T_y(Y)\). Note the existence of \(u \in T_x(X)\) such that \((dg_y \circ df_x)(u) + w = dg_y(\tilde{w})\), and moreover, \(dg_y(df_x(u - \tilde{w})) \in T_x(W)\) is guaranteed by \((g \circ f)\cap W\). □

**Exercise 34** (1.5.9). Let \(V\) be a vector space, \(\Delta\) the diagonal of \(V \times V\), \(A : V \to V\) linear map, and \(W = \Gamma(A)\). Then \(W \cap \Delta\) if and only if 1 is not an eigenvalue of 

Solution) Since \(\Delta\) and \(W\) are both vector subspaces of \(V \times V\), \(W \cap \Delta\) is equivalent to \(W + \Delta = V \times V\). Fix an arbitrary \((u, v) \in V \times V\), then if we can always find \((v_1, v_1) \in \Delta, (v_2, Av_2) \in W\) such that \(v_1 + v_2 = u, v_2 + Av_2 = v\) if and only if \((A - I)\) is invertible. □
**Exercise 35** (1.5.10). Let \( f : X \to X \) be a smooth map with fixed point \( x \) (i.e. \( f(x) = x \)). If \( 1 \) is not an eigenvalue of \( df_x : T_x(X) \to T_x(X) \), then \( x \) is called a **Lefschetz fixed point of** \( f \), and \( f \) is **Lefschetz map** if all its fixed points are Lefschetz. If \( X \) is compact and \( f \) is Lefschetz, then \( f \) has only finitely many fixed points.

Solution) Let \( \Delta_X \) be the diagonal of \( X \times X \), respectively. We wish to show that \( \Delta_X \cap \Gamma(f) \) is finite. **Claim:** it suffices to show \( \exists \Gamma(f) \). If it is so, then \( \Delta_X \cap \Gamma(f) \subset X \times X \) is a submanifold of dimension 0, since \( \dim \Delta_X = \dim \Gamma(f) = \dim X \) (\text{by Exercise 1.1.16,17}). Now, \( X \times X \) is compact so its 0-dimensional submanifold is finite (if not then it has a limit point, which does not admit a neighborhood diffeomorphic to a point).

Now we show \( \Delta_X \cap \Gamma(f) \). Fix any \( x \in \Delta_X \cap \Gamma(f) \), and let \( \Delta_Y \) be the diagonal of \( T_x(X) \times T_x(X) \). Then by [Exercise 1.5.9] we have \( T(x,x)(\Delta_X + T(x,x)(\Gamma(f))) = \Delta_Y + \Gamma(df_x) = T_x(X) \times T_x(X) = T_x(X \times X) \) (first equality by [Exercise 1.2.10,10]). \( \Box \)

### 1.6 Homotopy and Stability

**Exercise 36** (1.6.1,2). Suppose \( f_0 \sim f_1 \) (homotopic), then \( \exists \tilde{F} : X \times I \to Y \) homotopy such that \( \tilde{F}(x,t) = f_0(x) \ \forall t \in [0,1/4] \) and \( \tilde{F}(x,t) = f_1(x) \ \forall t \in [3/4,1] \). Hence, homotopy is an equivalence relation.

Solution) By [Exercise 1.1.18], there exists a function \( h : \mathbb{R} \to \mathbb{R} \) such that \( h(x) = 0 \) on \( x \leq 1/4 \) and \( h(x) = 1 \) on \( x \geq 3/4 \). Now, if \( F \) is the homotopy between \( f_0, f_1 \), set \( \tilde{F} = F \circ (\text{Id} \times h) \).

Now we show equivalence relation. Let \( F,G \) be homotopies for \( f \sim g, g \sim h \). Then define \( H : X \times [0,7/4] \to Y \) by

\[
H(x,t) = \begin{cases} 
\tilde{F}(x,t) & t \in [0,1] \\
\tilde{G}(x,t-3/4) & t \in [3/4,7/4] 
\end{cases}
\]

Then \( H \) is smooth, and \( \tilde{H} := H(x, \frac{4}{7}t) \) is the desired homotopy for \( f \sim h \). \( \Box \)

**Exercise 37** (1.6.3). Every connected manifold \( X \) is **arcwise connected**, i.e. \( \forall x_0, x_1 \in X, \exists f : I \to X \) smooth with \( f(0) = x_0, f(1) = x_1 \).

Solution) We first note that arcwise connectedness \( (x_0 \sim x_1) \) is an equivalence condition since \( x_0 \sim x_1 \Leftrightarrow f_0 \sim f_1 \) where \( f_0 : \{\ast\} \to X, \ast \mapsto x_0, f_1 : \{\ast\} \to X, \ast \mapsto x_1 \), and homotopy is an equivalence relation. By going through the local parametrizations, it is easy to show that arcwise connected components are clopen in \( X \), and hence if \( X \) is connected it is arcwise connected. \( \Box \)

**Exercise 38** (1.6.7). The antipodal map \( \alpha : S^k \to S^k, x \to -x \) is homotopic to the identity if \( k \) is odd.

Solution) Note that for any fixed \( \theta \in \mathbb{R} \), the map \( L_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \) given by multiplying the matrix

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta 
\end{bmatrix}
\]

preserves the norm: i.e. \( \| (x,y) \| = \| L(x,y) \| \). So, for \( S^{2k-1} \subset \mathbb{R}^{2k} \), the map \( L := L_{\pi t} \times \ldots \times L_{\pi t} : \mathbb{R}^{2k} \times I \to \mathbb{R}^{2k} \) is the smooth homotopy. \( \Box \)
Exercise 40 (1.6.9). Let $\rho : \mathbb{R} \to \mathbb{R}$ be a function with $\rho(s) = 1$ if $|s| < 1$, $\rho(s) = 0$ if $|s| > 2$, and define $f_t : \mathbb{R} \to \mathbb{R}$ by $f_t(x) = x\rho(tx)$. This is a counterexample to all the parts of the stability theorem when $X$ is not compact.

Solution) We simply need verify. $f_0(x) = x\rho(0) = x$, hence $f_0$ is a diffeomorphism and any submanifold of $\mathbb{R}$ is transversal to identity. Now, for any $t > 0$, note that $|tx| > 2$ for all $|x| > 2/|t|$, so $f_t$ cannot be local diffeomorphism/immersion/submersion/embedding/diffeomorphism. And for $|x| > 2/|t|$, $f_t(x) = 0$, so clearly, $\{0\}$ is not transversal to $f_t$. □

Exercise 41 (1.6.10). A deformation of a submanifold $Z \subset Y$ is a smooth homotopy $i_t : Z \to Y$ where $i_0$ is the inclusion map and each $i_t$ is an embedding (and thus, $Z_t = i_t(Z)$ is a smoothly varying submanifold of $Y$ with $Z_0 = Z$). If $Z$ is compact, then any homotopy $i_t$ of its inclusion map is a deformation for small $t$.

Solution) Since $Z$ is compact, embedding is stable class. □

1.7 Sard’s Theorem and Morse Functions

Exercise 42 (1.7.5). Exhibit a smooth map $f : \mathbb{R} \to \mathbb{R}$ whose set of critical values is dense.

Solution) From [Exercise 1.1.18], there is a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 1$ if $|x| \leq 1/4$ and $g(x) = 0$ if $|x| \geq 1/2$. Now, write $\mathbb{Q} = \{q_1, q_2, \ldots\}$, and then for $i \in \mathbb{N}$, define $g_i : \mathbb{R} \to \mathbb{R}$ by $g_i(x) = q_ig(x - i)$. Now define $f := \sum_i g_i$, then all the rationals are critical values for $f$ and dense in $\mathbb{R}$.

Remark: Measure zero implies empty interior, but the converse is false. □

Exercise 43 (1.7.6). The sphere $S^k$ is simply connected if $k > 1$.

Solution) Since $\dim S^1 < \dim S^k$ ($k > 1$), $p \in S^k$ is a regular value iff $p \notin f(S^1)$. That is, Sard’s Theorem implies that $f(S^1)$ is measure zero in $S^k$, hence $\exists p \notin f(S^1)$. Now under $S^k - \{p\} \simeq \mathbb{R}^k$, and with $\mathbb{R}^k$ being contractible, $f(S^1)$ is homotopic to a constant map. □

Exercise 44 (1.7.8). Analyze the critical behavior at the origin in the following functions:

a. $f(x, y) = x^2 + 4y^3$

b. $f(x, y) = x^2 - 2xy + y^2$
c. $f(x, y) = x^2 + y^4$

d. $f(x, y) = x^2 + 11xy + y^2/2 + x^6$

e. $f(x, y) = 10xy + y^2 + 75y^3$

Solution) In the order of nondegenerate?/isolated?/local min or max?(non strict)?, we have:

a. N/N/N, b. N/N/Y(min), c. N/Y/Y(min), d. Y/Y/N, e. Y/Y/N □

**Exercise 45** (1.7.11, 12). If $a \in \mathbb{R}^n$ is a non degenerate critical point of $f : \mathbb{R}^n \to \mathbb{R}$, there exists a local coordinate system $(x_1, \ldots, x_n)$ around $a$ such that

\[ f = f(a) + \sum_{i=1}^{n} \epsilon_i x_i^2, \quad \epsilon_i = \pm 1 \]

And hence derive the usual second derivative test.

Solution) Note that Hessian matrix is always real symmetric and hence diagonalizable by orthogonal matrix $P$. Let $H$ be Hessian matrix of $f$ at $a$, and $P$ be the orthogonal matrix that diagonalizes $H$. Morse Lemma gives us local coordinates $x^\prime := (x_1^\prime, \ldots, x_n^\prime) : \mathbb{R}^n \to \mathbb{R}^n$ around $a$ such that $f = f(a) + x^\prime H x^\prime$. Then $x^\prime := (x_1^\prime, \ldots, x_n^\prime) := P^T x^\prime$ is a new local coordinates around $a$ such that $f = f(a) + x^\prime P^T H P P^T x^\prime = f(a) + x^\prime (P^T H P) x^\prime$. Then,

\[ f = f(a) + \sum_{i=1}^{n} \lambda_i x_i^2, \quad \lambda_i \text{’s are eigenvalues of } H \]

Finally, let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $(y_1, \ldots, y_n) \mapsto (\sqrt{\lambda_1} |y_1|, \ldots, \sqrt{\lambda_n} |y_n|)$, and $x := (x_1, \ldots, x_n) := g \circ x^\prime$ be the new local coordinates around $a$. Then

\[ f = f(a) + \sum_{i=1}^{n} \epsilon_i x_i^2, \quad \epsilon_i = \pm 1 \]

as desired, and since $(x_1, \ldots, x_n)(a) = 0$ the second derivative test immediately follows. □

**Exercise 46** (1.7.14). Check that the "height function" $(x_1, \ldots, x_k) \mapsto x_k$ on the sphere $S^{k-1}$ is a Morse function with two critical points, the poles (one max, one min).

Solution) Let $f : S^{k-1} \to \mathbb{R}$ be the restriction of the projection $\pi : \mathbb{R}^k \to \mathbb{R}$, $(x_1, \ldots, x_k) \mapsto x_k$. Then for $x \in S^{k-1}$, $df_x : T_x(S^{k-1}) \to \mathbb{R}$ is the restriction of $d\pi_x = \pi$. Thus, $x \in S^{k-1}$ is a critical value iff $T_x(S^{k-1}) = \mathbb{R}^{k-1} \times \{0\}$. Since, $S^{k-1} = g^{-1}(1)$ where $g : x \mapsto \|x\|^2$, $T_a(S^{k-1}) = \ker(x \mapsto 2a^\top x)$, $T_a(S^{k-1}) = \mathbb{R}^{k-1} \times \{0\}$ exactly when $a = (0, \ldots, 0, 1) := N$ or $(0, \ldots, 0, -1) := S$. Now, $\phi_{\pm} : \mathbb{R}^{k-1} \supset B_1(0) \to \mathbb{R}$, $x \mapsto x \times \pm \sqrt{1 - \|x\|^2}$ are local parametrizations of $N, S$. And calculating $H(f \circ \phi_{\pm})_0$, we have $-I$ (hence max), and for the $-$ case we have $I$.

**Exercise 47** (1.7.16). Let $U \subset \mathbb{R}^k$ open, $f : U \to \mathbb{R}$ smooth, and $H(x)$ be the Hessian of $f$ for $x \in U$. Then $f$ is Morse if and only if

\[ \det(H)^2 + \sum_{i=1}^{k} \left( \frac{\partial f}{\partial x_i} \right)^2 > 0 \text{ on } U \]
Solution) $x \in U$ is a critical point of $f$ iff $\nabla f = 0$, and hence iff $\sum_{i=1}^{k} \left( \frac{\partial f}{\partial x_i} \right)^2 = 0$. The rest is follows immediately from definition of Morse. □

**Exercise 48 (1.7.17,18). (Stability of Morse Functions)** Suppose $f_t$ is a homotopic family of functions on $\mathbb{R}^k$. Show that if $f_0$ is Morse in some neighborhood $f$ a compact set $K$, then so is every $f_t$ for $t$ sufficiently small. And thus, Morse function is stable class.

Solution) Let $U \subset \mathbb{R}^k$ be open containing $K$ such that $f_0 : U \to \mathbb{R}$ is Morse. Now, denote the Hessian of $f_t$ at $x$ by $Hf_t|_x$, and define $h : \mathbb{R}^k \times I \to \mathbb{R}$ by $(x,t) \mapsto \det(Hf_t|_x)^2 + \sum_{i=1}^{k} \left( \frac{\partial f_t}{\partial x_i} \right)^2$.

Clearly, $h$ is smooth, and by [Exercise 1.7.16], we know that $h > 0$ on $U \times \{0\}$, hence on $K \times \{0\}$. Since $K \times \{0\}$ is compact, $h \geq 2\delta$ for some $\delta > 0$. By continuity of $h$, there exists an open set $U' \supset K \times \{0\}$ such that $h > \delta$ on $U'$. By Tube Lemma, $\exists \epsilon > 0$ such that $h > \delta$ on $K \times [0,\epsilon]$. Using continuity of $h$ again, for any fixed $t \in [0,\epsilon]$ there exists open $V \supset K$ such that $h > 0$ on $V \times \{t\}$, as desired.

Now let $X$ be a compact manifold, $f_0 : X \to \mathbb{R}$ Morse, and $f_t$ homotopic family of functions. Suppose for any $x \in X$, there exists $U_x$, neighborhood of $x$, and $\epsilon_x > 0$ such that $f_t$ is Morse on $U_x$ for $t \in [0,\epsilon_x]$; then $\bigcup_x U_x$ form an open over of $X$, so choosing a finite sub cover $U_{x_1} \cup \cdots \cup U_{x_n}$, we have that $f_t$ is Morse on $X$ for any $t \in [0, \min(\epsilon_{x_1}, \ldots, \epsilon_{x_n})]$. Existence of $U_x$ and $\epsilon_x$ is given by [Exercise 1.7.17]: for $\phi : V_x \to X$ local parametrization around $x$ with $\phi(0) = x$, set $U_x = \phi(B_\epsilon(0))$ where $B_\epsilon(0) \subset V_x$. □

### 1.8 Embedding Manifolds in Euclidean Space

A few practice with tangent bundles:

**Exercise 49 (1.5.2).** Let $g : X \to \mathbb{R}$ be smooth, everywhere-positive. Then the map $f : T(X) \to T(X)$ given by $(x,v) \mapsto (x,g(x)v)$ is smooth.

Solution) If $m : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is scalar multiplication, $\Delta : \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ diagonal map, then $f$ is just restriction of $(\text{Id} \times m) \circ (\text{Id} \times g \times \text{Id}) \circ (\Delta \times \text{Id})$. □

**Exercise 50 (1.5.3,4).** $T(X \times Y)$ is diffeomorphic to $T(X) \times T(Y)$. $T(S^1)$ is diffeomorphic to $S^1 \times \mathbb{R}$.

Solution) First statement is immediate from $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$. The map $(x,v) \mapsto (x,||v||)$ is a diffeomorphism of $T(S^k)$ and $S^1 \times \mathbb{R}$. □

**Exercise 51 (1.5.6).** A vector field $\vec{v}$ on a manifold $X$ in $\mathbb{R}^N$ is a smooth map $\vec{v} : X \to \mathbb{R}^N$ such that $\vec{v}(x) \in T_x(X)$. Equivalently, a vector field $\vec{v}$ on $X$ is a cross section of $T(X)$—i.e. a smooth map $\vec{v} : X \to T(X)$ such that $p \circ \vec{v}$ is identity on $X$. Lastly, $x \in X$ is zero of a vector field $\vec{v}$ if $\vec{v}(x) = 0$.

Solution) (Just definitions)
Exercise 52 (1.5.7,8). If \( k \) is odd, there exists a nonvanishing vector field \( \vec{v} \) on \( S^k \). If \( S^k \) has a nonvanishing vector field \( \vec{v} \), then its antipodal map is homotopic to the identity.

Solution) If \( k \) is odd, then the map \( (x_1, \ldots, x_{k+1}) \mapsto (-x_2, x_1, -x_4, x_3, \ldots, -x_{k+1}, x_k) \) is a (linear) smooth map, i.e. a nonvanishing vector field on \( S^k \). Now, let \( \vec{v} \) is a nonvanishing vector field on \( S^k \), and WLOG \( \|\vec{v}(x)\| = 1 \), \( \forall x \in S^k \) (define new vector field by \( x \mapsto \vec{v}(x)/\|\vec{v}(x)\| \)), and thus \( \vec{v} : S^k \to S^k \) smooth and \( x \perp \vec{v}(x) \). Now, \( H : S^k \times I \to S^k \) defined by \( (x, t) \mapsto x \cos \pi t + \vec{v}(x) \sin \pi t \) is a homotopy between the identity and antipodal map. \( \square \)
2 Chapter 2. Transversality and Intersection

2.1 Manifolds with Boundary

Exercise 53 (Made-up). Let \( f : X \to Y \) be a smooth map of manifolds with boundary, \( Z \subset X \) a submanifold with boundary, and define \( g := f|_Z \). Then for \( x \in Z \), \( dg_x : T_x(Z) \to T_{f(x)}(Y) \) is equal to \( df_x|_{T_x(Z)} \).

Solution) The proof reads exactly like [Exercise 17]. \( \square \)

Exercise 54 (2.1.2). Let \( f : X \to Y \) be a diffeomorphism of manifolds with boundary. Then \( \partial f \) maps \( \partial X \) diffeomorphically onto \( \partial Y \).

Solution) A diffeomorphism \( f : X \to Y \) is locally equivalent to \( U \to V \) where \( U \subset \mathbb{R}^k \). So, it follows that \( \partial f(x) \in \partial Y \) if and only if \( x \in \partial X \). \( \square \)

Exercise 55 (2.1.3). \( S := [0,1] \times [0,1] \) is not a manifold with boundary.

Solution) Suppose \( S \) is a manifold with boundary, then \( s = (0,0) \in S \) has a neighborhood \( U \subset \mathbb{R}^2 \) diffeomorphic to \( V \subset \mathbb{R}^2 \). Let \( f : U \to V \) be the diffeomorphism, and shrinking if \( U \) is necessary, let \( F : \tilde{U} \to \mathbb{R}^2 \) be the smooth extension of \( f \) where \( U \subset \tilde{U} \subset \mathbb{R}^2 \). Then \( dF_s \) is an isomorphism.

Now, note that \( \partial U \) maps to \( \partial \mathbb{R}^2 \), so if \( F = (F_1,F_2) \) then \( F_2(x,0) = 0 = F_2(0,y) \) for any \( (x,0),(0,y) \in U \). Thus, \( \frac{\partial F_2}{\partial x}(s) = 0 \) an \( \frac{\partial F_2}{\partial y}(s) = 0 \). But this implies that \( dF_s(e_1), dF_s(e_2) \in \mathbb{R} \times \{0\} \) and thus not linearly dependent. \( \square \)

Exercise 56 (2.1.4). The solid hyperboloid defined by \( x^2 + y^2 - z^2 \leq a \) is a manifold with boundary.

Solution) Define \( \pi : \mathbb{R}^3 \to \mathbb{R} \) by \( (x,y,z) \mapsto a - (x^2 + y^2 - z^2) \). Since \( a > 0 \), it is easily checked that \( 0 \) is regular value of \( \pi \). Hence, \( \pi^{-1}((0,\infty)) \) is a manifold with boundary. \( \square \)

Exercise 57 (2.1.7). Let \( X \) be a manifold with boundary, \( x \in \partial X \), \( \phi : U \to X \) local parametrization with \( \phi(0) = x \) (so \( d\phi_0 : \mathbb{R}^k \to T_x(X) \) is isomorphism). We define the upper half space \( H_x(X) \) in \( T_x(X) \) by \( H_x(X) := d\phi_0(H^k) \). \( H_x(X) \) is independent of the choice of parametrization.

Solution)

Exercise 58 (2.1.8).

Solution)